

Fine-grained lower limit of entropic uncertainty in the presence of quantum memory

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The limitation on obtaining precise outcomes of measurements performed on two non-commuting observables of a particle as set by the uncertainty principle in its entropic form, can be reduced in the presence of quantum memory. We derive a new entropic uncertainty relation based on fine-graining, which leads to an ultimate limit on the precision achievable in measurements performed on two incompatible observables in the presence of quantum memory. We show that our derived uncertainty relation tightens the lower bound set by entropic uncertainty for members of the class of two-qubit states with maximally mixed marginals, while accounting for the recent experimental results using maximally entangled pure states and mixed Bell-diagonal states. An implication of our uncertainty relation on the security of quantum key generation protocols is pointed out.

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In the absence of quantum memory, the Heisenberg uncertainty principle[1] bounds the product of uncertainties, i.e., the spread measured by standard deviation, of measurement outcomes for two non-commuting observables. The Heisenberg uncertainty principle, for two observables R and S , is given by

$$\Delta R \Delta S \geq \frac{1}{2} |\langle [R, S] \rangle| \quad (1)$$

where ΔR (ΔS) represents the standard deviation which is a measure of uncertainty of the corresponding observable R (S). The possibility of violating the uncertainty principle using quantum entanglement was one of the off-shoots of the famous Einstein-Podolsky-Rosen argument[2]. An experiment to demonstrate the violation of the uncertainty principle was proposed by Popper[3], and subsequently realized much later by Kim and Shih[4]. Other experiments using entangled states to demonstrate the violation of the Heisenberg uncertainty principle have also been performed[5–7].

There is an increasing appreciation in recent times of the limitations of the use of standard deviation as a measure of uncertainty[8]. One of the drawbacks of the uncertainty relation in terms of standard deviation is that the right hand side of the inequality (1) depends on the state of the quantum system. To improve this situation as well as link uncertainty with information theoretic concepts, the uncertainty relating to the outcomes of observables has been reformulated in terms of Shannon entropy[9] instead of standard deviation. Entropic uncertainty relations for two observables was first introduced by Deutsch[10], following which an improved version given by

$$\mathcal{H}(R) + \mathcal{H}(S) \geq \log_2 \frac{1}{c} \quad (2)$$

was first conjectured[11], and then proved[12]. Here $\mathcal{H}(i)$ denotes the Shannon entropy of the probability distribution of the measurement outcomes of observable

i ($i \in \{R, S\}$) and $\frac{1}{c}$ quantifies the complementarity of the observable. For non-degenerate observables, $c = \max_{i,j} |\langle a_i | b_j \rangle|^2$, where $|a_i\rangle$ and $|b_j\rangle$ are eigenvectors of R and S , respectively.

In a recent work, Berta et al. [13] have shown that the lower bound of entropic uncertainty (given by Eq.(2)) can be improved in the presence of quantum memory, making use of the quantum information contained in the correlated state of the particle on which the two observables are measured, with the state of another particle. Specifically, the sum of uncertainties of two measurement outcomes ($\mathcal{H}(R) + \mathcal{H}(S)$) for measurement of two observables (R, S) on the quantum system (“A”, possessed by Alice) can be reduced to 0 (i.e., there is no uncertainty) if that system is maximally entangled with another system, called quantum memory (“B”, possessed by Bob). Here, Bob is able to reduce his uncertainty about Alice’s measurement outcome with the help of communication from Alice regarding the choice of her measurement performed, but not its outcome. The entropic uncertainty relation in the presence of quantum memory[13] is given by

$$\mathcal{S}(R|B) + \mathcal{S}(S|B) \geq \log_2 \frac{1}{c} + \mathcal{S}(A|B) \quad (3)$$

where $\mathcal{S}(R|B)$ ($\mathcal{S}(S|B)$) is the conditional von Neumann entropy of the state given by $\sum_j (|\psi_j\rangle\langle\psi_j| \otimes I) \rho_{AB} (|\psi_j\rangle\langle\psi_j| \otimes I)$, with $|\psi_j\rangle$ being the eigenstate of observable R (S), and $\mathcal{S}(R|B)$ ($\mathcal{S}(S|B)$) quantifies the uncertainty corresponding to the measurement R (S) on the system “A” given information stored in the system “B” (i.e., quantum memory). $\mathcal{S}(A|B)$ quantifies the amount of entanglement between the quantum system possessed by Alice and the quantum memory possessed by Bob. For example, if Alice and Bob share a maximally entangled state, $\mathcal{S}(A|B) = -1$, and for a two qubit case $\log_2 \frac{1}{c}$ can not larger than 1, and hence the right hand side of equation (3) cannot be greater than 0 for a maximally entangle state. It follows that for maximally entangled

state Bob's uncertainty of Alice's measurement outcome reduces to zero when Alice measures the same observable as Bob does on his quantum memory, and communicates with Bob about her measurement choice.

The effectiveness of quantum memory in reducing quantum uncertainty has been demonstrated in two recent experiments using respectively, pure[14] and mixed[15] entangled states. For the purpose of experimental verification of inequality (3), The entropic uncertainty is recast in the form of the sum of the Shannon entropies $\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S)$ when Alice and Bob measure the same observables $R(S)$ on their respective systems and get different outcomes whose probabilities are denoted by $p_d^R(p_d^S)$, and $\mathcal{H}(p_d^{R(S)}) = -p^{R(S)} \log_2 p^{R(S)} - (1 - p^{R(S)}) \log_2 (1 - p^{R(S)})$. Making use of Fano's inequality[16], it follows that $\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \mathcal{S}(R|B) + \mathcal{S}(S|B)$ which using Eq.(3) gives [15]

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \log_2 \frac{1}{c} + \mathcal{S}(A|B) \quad (4)$$

The right hand side of the inequality (4) can be determined from the knowledge of the state and the measurement settings. It was experimentally observed by Li et al. [15] that the left hand side exceeds the right hand side for the case of a Bell-diagonal state. It may be noted that the lower bound of entropic uncertainty given by the right hand sides of the relations (2) and (3) contain the term $1/c$ which depends on the choice of measurement settings.

A further improvement in the manifestation of the uncertainty in measurement outcomes has been motivated by the realization that entropic functions provide a rather coarse way of measuring the uncertainty of a set of measurements, as they do not distinguish the uncertainty inherent in obtaining any combination of outcomes for different measurements. In the same year of the work by Berta et al. [13], a new form of the uncertainty relation, *viz.*, *fine grained uncertainty relation*, was proposed by Oppenheim and Wehner [17]. In particular, they considered a game according to which Alice and Bob both receive binary questions, i.e., projective spin measurements along two different directions at each side. The winning probability is given by the relation [17]

$$P^{game}(\mathcal{T}_A, \mathcal{T}_B, \rho_{AB}) = \sum_{s,t} p(t_A, t_B) \sum_{a,b} V(a, b|t_A, t_B) \langle (A_{t_A}^a \otimes B_{t_B}^b) \rangle_{\rho_{AB}} \leq P_{max}^{game} \quad (5)$$

where ρ_{AB} is a bipartite state shared by Alice and Bob, and \mathcal{T}_A and \mathcal{T}_B represent the set of measurement settings $\{t_A\}$ and $\{t_B\}$ chosen by Alice and Bob, respectively, with probability $p(t_A, t_B)$. Alice's (Bob's) question and answer are $t_A(t_B)$ and $a(b)$, respectively, with $A_{t_A}^a = \frac{I + (-1)^a A_{t_A}}{2}$ ($B_{t_B}^b = \frac{I + (-1)^b B_{t_B}}{2}$) being a measurement of the observable A_{t_A} (B_{t_B}). Here $V(a, b|t_A, t_B)$ is some function determining the winning condition of the game.

The necessary condition for fine-graining is to consider a particular outcome, or particular choice of the winning condition in a game. The winning condition is the essence of fine-graining, and as shown in Ref.[17], every game gives rise to an uncertainty relation, and vice-versa.

The winning condition corresponding to a special class of nonlocal retrieval games (CHSH game [17]) for which there exist only one winning answer for one of the two parties, is given by $V(a, b|t_A, t_B) = 1$, iff $a \oplus b = t_A \cdot t_B$, and 0 otherwise. P_{max}^{game} is the maximum winning probability of the game, maximized over the set of projective spin measurement settings $\{t_A\} (\in \mathcal{T}_A)$ by Alice, the set of projective spin measurement settings $\{t_B\} (\in \mathcal{T}_B)$ by Bob, i.e., $P_{max}^{game} = \max_{\mathcal{T}_A, \mathcal{T}_B, \rho_{AB}} P^{game}(\mathcal{T}_A, \mathcal{T}_B, \rho_{AB})$. Using the maximum winning probability it is possible to discriminate between classical theory, quantum theory and no-signaling theory with the help of the degree of nonlocality [17]. A generalization to the case of tripartite systems has also been proposed [18].

In the present work we derive a new form of the uncertainty relation in the presence of quantum memory, in which the lower bound of entropic uncertainty corresponding to the measurement of two observables is determined by fine-graining of the possible measurement outcomes, and is thus independent of the specific choice of measurement settings. We find the finer or optimized lower bound of entropic uncertainty, which represents the ultimate limit to which uncertainty of outcomes of two non-commuting observables can be reduced by performing any set of measurements in the presence of quantum memory. The new uncertainty relation derived by us is able to account for the two experimental results obtained for the case of maximally entangled states[14] and mixed Bell-diagonal states[15]. More interestingly, we show that when the quantum correlations are made using the class of two-qubit states with maximally mixed marginals, the fundamental limit set by our uncertainty relation prohibits the attainment of the lower bound of entropic uncertainty[13] as defined by the right hand side of equation (3). We further discuss the ramification of our uncertainty relation on an application for key extraction in quantum key generation[13, 20, 21].

We consider a quantum game played by Alice and Bob for which the winning probability is given by the fine-grained uncertainty relation [17]. In this game, Alice and Bob share a two-qubit state ρ_{AB} which is prepared by Alice, after which she sends one of the qubits to Bob. Bob's qubit represents the quantum memory, and mimicking the scenario of references [13–15] we look at Bob's uncertainty of the outcome of Alice's measurement of one of two incompatible observables (say, R and S), when Alice helps Bob by communicating her measurement choice of a suitable spin observable on her system. In this game Alice and Bob are driven by the requirement of minimizing the value of the quantity $\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S)$ which forms the left hand side of the entropic uncertainty relation (4).

The minimization is over all incompatible measurement settings such that $R \neq S$, i.e.,

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \min_{R, S \neq R} [\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S)] \quad (6)$$

To find the minimum value, the choice of one variable, e.g., R , may be fixed, without the loss of generality to be, say σ_z (spin measurement along the z -direction), and then the minimization be performed over the other variable S . Hence, Eq.(6) can be rewritten in the form

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \mathcal{H}(p_d^{\sigma_z}) + \min_{S \neq \sigma_z} [\mathcal{H}(p_d^S)] \quad (7)$$

The uncertainty defined by the entropy $\mathcal{H}(p_d^S)$ is minimum when the certainty of the required outcome is maximum, corresponding to an infimum value for the probability p_d^S .

Now, in order to obtain the infimum value of p_d^S , we use the fine-grained uncertainty relation [17] in a form relevant to the present situation. In the language of the above bipartite game, Alice and Bob measure the same observables (σ_z and S) on their respective systems, and win the game if their measurement outcomes, either 0 or 1 (a for Alice, and b for Bob) are correlated in the form $a \oplus b = 1$, i.e., they get different outcomes. Therefore, the fine-grained uncertainty relation (5) in the context of this particular game considered here ($s = t$ in Eq.(5)), will now be determined by a new winning condition as given below where the infimum value of the winning probability (corresponding to minimum uncertainty) is given by

$$p_{\text{inf}}^S = \inf_{S(\neq \sigma_z)} \sum_{a,b} V(a,b) \text{Tr}[(A_S^a \otimes B_S^b) \cdot \rho_{AB}], \quad (8)$$

with the winning condition $V(a,b)$ given by

$$\begin{aligned} V(a,b) &= 1 && \text{iff } a \oplus b = 1 \\ &= 0 && \text{otherwise.} \end{aligned} \quad (9)$$

with A_S^a being a projector for observable S with outcome ' a ', given by $S^\alpha = \frac{I + (-1)^\alpha \vec{n}_S \cdot \vec{\sigma}}{2}$ (and similarly for B_S^b), where $\vec{n}_S (\equiv \{\sin(\theta_S) \cos(\phi_S), \sin(\theta_S) \sin(\phi_S), \cos(\theta_S)\})$; $\vec{\sigma} \equiv \{\sigma_x, \sigma_y, \sigma_z\}$ are the Pauli matrices; α takes the value either 0 (for spin up projector) or 1 (for spin down projector). Note here that the above winning condition proposed by us is different from the winning condition used in ref. [17] for the purpose of capturing the nonlocality of bipartite systems. The essence of fine-graining is to consider a particular outcome, or particular choice of the winning condition in a game. We have adapted the fine-grained uncertainty relation making it directly applicable to the experimental situation of quantum memory, by introducing a new winning condition modelling the experiments [15].

The minimum value of uncertainty thus obtained by minimizing over all measurements is now substituted in

the second term of the right hand side of Eq.(7) from which the expression for the final form of our uncertainty relation

$$\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{\text{inf}}^S) \quad (10)$$

follows giving the optimal lower bound of entropic uncertainty. The value of p_{inf}^S is calculated for the given quantum state ρ_{AB} using the expression (8). As a result, the lower bound of the entropic uncertainty in the presence of quantum correlations is now determined by the minimum entropy corresponding to the infimum winning probability of the above game, replacing the earlier lower bound given by the right hand side of Eq.(4) [13, 15]. Note that the inequality (10) can be derived for any choice of R other than σ_z as well. Our proposed uncertainty relation is independent of the choice of measurement settings as it optimizes the reduction of uncertainty quantified by the conditional Shannon entropy over all possible observables. Given a bipartite state possessing quantum correlations, inequality (10) provides the fundamental limit to which uncertainty in the measurement outcomes of any two incompatible variables can be reduced.

In the following analysis, we illustrate the efficacy of our uncertainty relation (10) with some examples. We use Eq.(8) to first calculate the value of p_{inf}^S (the optimization over all spin projectors is performed using *Mathematica*) and the corresponding measurement setting S , and then use it to find the minimum value or lower bound of uncertainty defined by the right hand side of the equation (10) for the examples of the different states representing quantum memory discussed here, viz., maximally entangled state, Bell-diagonal state, an example from the class of two-qubit states with maximally mixed marginals, and the Werner state. We further describe how our derived uncertainty relation affects an important application to quantum key distribution[19] modifying the earlier bounds[20] on the amount of key per state that Alice and Bob are able to extract[13, 21]

First, we consider that Alice and Bob share a maximally entangled state. For any maximally entangled state the outputs are strongly correlated when both Alice and Bob measure the same observable on their respective systems. When Alice communicates about her measurement setting, Bob knows with certainty about Alice's outcome by measuring the same observable on his system. The lower bound of entropic uncertainty should thus reduce to zero in this case, as observed earlier[13, 14]. Using our uncertainty relation (10), it indeed follows that $\mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{\text{inf}}^S) = 0$, a result which holds for any choice of the observable S , as long as Alice and Bob measure the same observables. Now, for an observable parametrized by $S = \hat{n}_S \cdot \vec{\sigma}$, where $\hat{n}_S (\equiv \{\sin(\theta_S) \cos(\phi_S), \sin(\theta_S) \sin(\phi_S), \cos(\theta_S)\})$, note that the right hand side of the entropic uncertainty relation (4) is given by $\log_2 \frac{1}{c} + \mathcal{S}(A|B) = -1 +$

$\log_2[\frac{1}{\max[\cos(\frac{\theta_S}{2})^2, \cos(\frac{\theta_S}{2})^2]}]$ which varies from 0 to -1 depending upon the measurement settings $\{\theta_S, \phi_S\}$, and hence the lower bound of the uncertainty relation given by Berta et al. [13] for a maximally entangled state is 0 only when the observables R and S are complementary to each other, i.e., $c = \max_{i,j} |\langle a_i | b_j \rangle|^2 = \frac{1}{2}$, as observed experimentally with horizontal and vertical polarized photons by Prevedel et al.[14].

Next, let us consider the Bell-diagonal state (used in the experiment by Li et al. [15]) given by $\rho_m = p|\phi^+\rangle\langle\phi^+| + (1-p)|\psi^-\rangle\langle\psi^-|$, where, $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and p lies between 0 and 1. The lower bound of entropic uncertainty in the presence of quantum memory (given by inequality (4)) for the state ρ_m with the choice of observables $S = \sigma_x$ and $R = \sigma_z$ ($\theta_S = \pi/2, \phi_S = 0$) is given by $\mathcal{H}(p) \equiv \mathcal{H}(1-p)$. It can be verified that the lower bound of our uncertainty relation (10) is obtained for the same variables with the corresponding probabilities given by $p^{\sigma_z} = 1-p$ and $p_{inf}^S \equiv p^{\sigma_y} = 1$, leading to saturation of the bound $\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S) \geq \mathcal{H}(1-p)$ for the Bell-diagonal state. Note here that the choice $R = \sigma_z$ $S = \sigma_x$ (as taken by Li et al. [15]) is unable to minimize the left hand side of the above expression, and thus we account for the fact that their experimental result (left hand side of the inequality (4)) is obtained to lie above the lower bound.

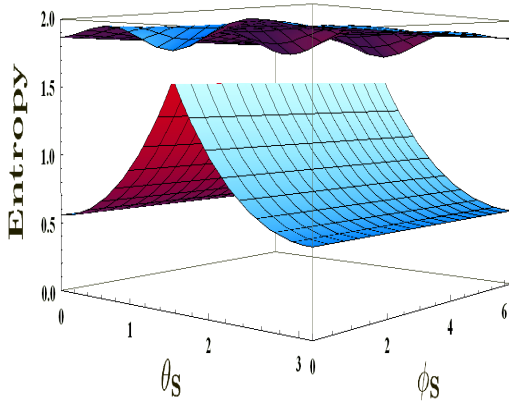


FIG. 1: Coloronline. The lower bound of entropic uncertainty corresponding to measurements on a two-qubit state with maximally mixed marginals in the presence of quantum memory. (i) the upper plot $\mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{inf}^S)$ as predicted by using our uncertainty relation (10) derived here, and (ii) the lower plot $\log_2 \frac{1}{c} + \mathcal{S}(A|B)$ as predicted by the analysis of Berta et al.[13] given by (3). The region between the two curves is inaccessible in actual measurements according to our results, since the optimal lower bound of entropic uncertainty is determined by fine-graining.

Finally, we consider the general class of two-qubit states with maximally mixed marginals, given by $\rho_{MM} = \frac{1}{4}(I_{4 \times 4} + \sum_{i=1}^3 c_i \sigma_i \otimes \sigma_i)$, where the c_i are real constants

satisfying the constraints $0 \leq \frac{1-c_1-c_2-c_3}{4} \leq 1, 0 \leq \frac{1+c_1-c_2-c_3}{4} \leq 1, 0 \leq \frac{1-c_1+c_2-c_3}{4} \leq 1, 0 \leq \frac{1-c_1-c_2+c_3}{4} \leq 1$, such that the state ρ_{MM} is physical. We obtain the lower bound of the inequality (10), and the corresponding setting S which is then used to compare the minimum of entropy thus obtained with the lower bound of the inequality (4). For a wide range of choices of the state parameters c_i we find that the fundamental limit set by the inequality (10) as obtained through fine-graining exceeds the lower bound applying the right hand side of equation (4). A typical example, using the values $c_1 = 0.5, c_2 = -0.2, c_3 = -0.3$, is illustrated in Fig.1. Note that the minimum value of $\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S)$ occurs in this case for $\theta_S = \pi/2, \phi_S = 0$, yielding the lower bound of (10), $\mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{inf}^S) \approx 1.745$, while for the same observables, one obtains the right hand side of (4) as $\log_2 \frac{1}{c} + \mathcal{S}(A|B) \approx 1.558$. It is seen from Fig.1 that when this specific state is used as quantum memory, the lower bound of entropic uncertainty as predicted by the analysis of Berta et al. [13] is never achievable in an actual experiment using any choice of the measurement settings $\{\theta_S, \phi_S\}$. As a further illustration of these results, one may also consider the Werner state $\rho_W = \frac{1-p}{4}I \otimes I + p|\psi^-\rangle\langle\psi^-|$. Here fine-graining leads to the lower bound $2\mathcal{H}(\frac{1+p}{2})$ which always exceeds the right hand side of the inequality (4) ($-3\frac{1-p}{4}\log_2 \frac{1-p}{4} - \frac{1+3p}{4}\log_2 \frac{1+3p}{4}$), except for $p = 0$ (maximally mixed state leading to maximum and equal uncertainty using both the approaches), and for $p = 1$ (maximally entangled state leading to vanishing uncertainty in both approaches).

The uncertainty principle in its entropic form could be used for verifying the security of key distribution protocols. It was derived by Devetak and Winter [20] that the amount of key K that Alice and Bob are able to extract per state should always exceed the quantity $\mathcal{S}(R|E) - \mathcal{S}(R|B)$, where the quantum state ρ_{ABE} is shared between Alice, Bob and the eavesdropper Eve (E)[19]. Extending this idea by incorporating the effect of shared quantum correlation between Alice and Bob, Berta et al.[13] reformulated their result (3) in the form of $\mathcal{S}(R|E) + \mathcal{S}(R|B) \geq \log_2 \frac{1}{c}$ conjectured earlier[21], enabling them to derive a new lower bound on the key extraction rate, given by $K \geq \log_2 \frac{1}{c} - \mathcal{S}(R|B) - \mathcal{S}(S|B)$. Now, using our uncertainty relation (10), it is possible to obtain a tighter lower bound, given by $K \geq \log_2 \frac{1}{c} - \min_{R,S} [\mathcal{H}(p_d^R) + \mathcal{H}(p_d^S)]$ which reduces to the form $K \geq \log_2 \frac{1}{c} - \mathcal{H}(p_d^{\sigma_z}) + \mathcal{H}(p_{inf}^S)$ when Alice and Bob retain data whenever they make the same choice of measurement on their respective sides. Note that the bound derived here is upper-bounded by the result of Berta et al.[13]. The implication is that the saturation of the bound derived earlier[13] is not possible for all states, and the bound derived here represents the optimal lower limit of key extraction valid for any shared correlation,

and for all measurement settings used by Alice and Bob.

To summarize, in the present work we give the optimized lower bound of entropic uncertainty in the presence of quantum memory [13] with the help of the fine-grained uncertainty principle [17], thus providing a new manifestation of observer dependence [22] of the fundamental limitation. Since entropy (or uncertainty) is directly related to probability, the purpose of minimizing probability as we have done while implementing the fine-graining, is essential to minimize uncertainty. So, we are able in this way to obtain the optimal lower bound of entropic uncertainty in presence of quantum memory. In measurements and communication involving two parties, the lower bound of entropic uncertainty cannot fall below the bound derived here, as we illustrate with several examples. Our uncertainty relation is independent of measurement settings, providing an operationally relevant fundamental limitation on the precision of outcomes for measurement of two incompatible observables in the presence of quantum memory. Implications on information processing exist, as discussed for the issue of privacy of quantum key generation.

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